PMM U.S.S.R.,Vol.50,No.5,pp.663-666,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

A FAMILY OF ALMOST CIRCULAR ORBITS IN THE INNER VERSION OF THE CIRCULAR THREE-BODY PROBLEM*

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A class of periodic orbits in the classical restricted three-body problem is determined. A version of the small-parameter method is used. The convergence of the solutions obtained is proved and their domain of convergence is shown.

1. Formulation of the problem. Let a material point P_0 of mass m_0 roatate with constant angular velocity n round a circular Keplerian orbit about a point P_1 of mass m_1. We take the point P_1 as the origin of the rectangular $OXYZ$ system of coordinates. We take the plane of the orbit of point P_0 as the basic coordinate XOY plane, and the straight line P_1OP_0 as the OX axis. We choose the positive direction of the OY axis so that the averaged motion is positive.

Now let a passively gravitating third body P rotate about P_0. Then its equations of motion will be written as /1/

$$X^{\cdot\cdot} - 2nY^{\cdot} - n^2X = \frac{\partial U}{\partial X}, \quad Y^{\cdot\cdot} - 2nX - n^2Y = \frac{\partial U}{\partial Y}, \quad Z^{\cdot\cdot} = \frac{\partial U}{\partial Z} \tag{1.1}$$

$$U = f\left(\frac{m_0}{R} + \frac{m_1}{R_0}\right) + \frac{n^2 m_1 X_0 X}{m_0 + m_1}$$

$$R^2 = X^2 + Y^2 + Z^2, \quad R_0^2 = (X - X_0)^2 + Y^2 + Z^2$$

(X_0 is the abscissa of the point P_0 and f is the gravitational constant).

Expanding R_0 in powers of X/X_0 and making the substitution /2/

$$X = \alpha x, \quad Y = \alpha y, \quad Z = \alpha z, \quad t = k\tau/\sqrt{fm_1}$$

with the assumption that

$$k = \alpha^{3/2}(k_0 + \alpha k_1 + \ldots), \quad n/\sqrt{fm_1} = \alpha^{-3/2}(n_0 + \alpha n_1 + \ldots)$$

we reduce system (1.1) to the form

$$x'' - 2k_0n_0y' - k_0^2n_0^2x + \frac{k_0^2x}{r^3} + \alpha F_1 = \alpha^2\frac{\partial U}{\partial x} \tag{1.2}$$

$$y'' + 2k_0n_0x' - k_0^2n_0^2y + \frac{k_0^2y}{r^3} + \alpha F_2 = \alpha^2\frac{\partial U}{\partial y}$$

$$z'' + \frac{k_0^2z}{r^3} + \alpha F_3 = \alpha^2\frac{\partial U}{\partial z}$$

where

$$F_1 = Ay' + (B + \chi)x, \quad F_2 = -Ax' + (B + \chi)y, \quad F_3 = \chi z$$

$$A = 2(k_0n_1 + k_1n_0), \quad B = 2(k_0k_1n_0^2 + k_0^2n_0n_1), \quad \chi = -\frac{2k_0k_1}{r^3}$$

$$\frac{\partial U}{\partial x} = -\frac{m_0k_0^2}{m_1x_0^2} + O(\alpha), \quad \frac{\partial U}{\partial y} = O(\alpha), \quad \frac{\partial U}{\partial z} = O(\alpha)$$

2. Proof of the existence of periodic solutions. Let us change to new variables using the relations

$$x = \rho\cos(v - k_0n_0\tau), \quad y = \rho\sin(v - k_0n_0\tau), \quad z = \zeta$$

Then system (1.2) can be rewritten in the form

$$\rho'' - \rho(v')^2 + \frac{k_0^2}{r^3}\rho + \alpha F_1 = \alpha^2\frac{\partial U}{\partial \rho} \tag{2.1}$$

$$\frac{d}{d\tau}(\rho^2v') + \alpha F_2 = \alpha^2\frac{\partial U}{\partial v}, \quad \zeta'' + \frac{k_0^2}{r^3}\zeta + \alpha F_3 = \alpha^2\frac{\partial U}{\partial \zeta}$$

When $\alpha = 0$, the system admits of a particular solution $\rho = 1$, $v = k_0\tau$, $\zeta = 0$ which determines the circular Keplerian motion of a passively gravitating material point.

*Prikl.Matem.Mekhan.,50,5,856-859,1986

Let us introduce the new unknown functions

$$\rho = 1 + \xi, \quad v = \eta + k_0\tau, \quad \zeta = \xi$$

for which system (2.1) will have the form

$$\begin{aligned}
\xi'' &= (1+\xi)(k_0+\eta') - \frac{k_0^2}{r^3}(1+\xi) - \alpha F_1 + \alpha^2 \frac{\partial U}{\partial \xi} \\
\eta'' &= \frac{2(k_0+\eta')\xi'}{1+\xi} + \frac{\alpha F_2}{1+\xi} + \frac{\alpha^2}{(1+\xi)^2}\frac{\partial U}{\partial \eta} \\
\zeta'' &= -\frac{k_0^2}{r^3}\zeta + \alpha F_3 + \alpha^2 \frac{\partial U}{\partial \zeta}
\end{aligned} \tag{2.2}$$

We shall seek the solution of the last system in the form of series in powers of the small parameter

$$\xi = \sum_{i=2}^{\infty} \alpha^i \xi_{i-1}, \quad \eta = \sum_{i=2}^{\infty} \alpha^i \eta_{i-1}, \quad \zeta = \sum_{i=2}^{\infty} \alpha^i \zeta_{i-1} \tag{2.3}$$

Let us substitute (2.3) into (2.2) and equate terms accompanying the like powers of α on the right-hand and left-hand sides of the last system of equations. Now let $k_0 n_1 + k_1 n_0 = k_0 k_1$. Then the general solution of the system of equations for ξ_1, η_1, ζ_1 will have the form

$$\begin{aligned}
\xi_1 &= \beta_1 \cos k_0\tau + \beta_2 \sin k_1\tau + \beta_3 + a_{11}\cos k_0 n_1 \tau \\
\eta_1 &= -2\beta_1 k_0 \tau + 2\beta_2 \cos k_1 \tau + b_{01}\tau + b_{11}\sin k_0 n_1 \tau + \beta_6 \\
\zeta_1 &= \beta_4 \cos k_0 \tau + \beta_5 \cos k_0 \tau \\
a_{11} &= \frac{k_0^2 (n_0+1)}{x_0^2 (n_0-1)(n_0-2) n_0}, \qquad b_{01} = \frac{3}{2} k_0 \beta_3 \\
b_{11} &= \frac{k_0}{x_0^2 (n_0-1)} \left[\frac{1}{2k_0} - \frac{(n_0^2+2n_0+7)(n_0+1)k_0}{2(n_0-1)(n_0-2)n_0} \right]
\end{aligned}$$

($\beta_1 \ldots \beta_6$ are arbitrary constants of integration). We shall assume that $n_0 = p/q$ is not an integer $(p, q = \pm 1, \pm 2, \ldots)$. Then the right-hand sides of the system of Eqs.(2.2) will be periodic functions with period $T = 2\pi k_0^{-1} q$.

We shall require that the following conditions hold:

$$\begin{aligned}
&\psi_1 = \xi(T) - \xi(0) \equiv 0, \quad \psi_2 = \xi'(T) - \xi'(0) \equiv 0, \quad \psi_3 = \eta(T) - \\
&\quad \eta(0) \equiv 0 \\
&\psi_4 = \eta'(T) - \eta'(0) \equiv 0, \quad \psi_5 = \zeta(T) - \zeta(0) \equiv 0, \quad \psi_6 = \zeta'(T) - \\
&\quad \zeta'(0) \equiv 0
\end{aligned}$$

from which we can find $\beta_1, \ldots, \beta_6$ as single-valued functions of α. Then ξ, η, ζ will be periodic functions with a common period T. The conditions of periodicity for system (2.2) will take the form

$$\begin{aligned}
&\psi_1/\alpha = \beta_1 \cos k_0 T - \beta_1 + \beta_2 \sin k_0 T + a_{11}\cos k_0 n_1 T + O(\alpha) \equiv 0 \\
&\psi_2/\alpha = -k_0\beta_1 \sin k_0 T + k_0 \beta_2 \cos k_0 T - k_0 \beta_2 + a_{11} k_0 n_1 \sin k_0 n_1 T + \\
&\quad O(\alpha) \equiv 0, \quad \psi_3/\alpha = -2\beta_1 \sin k_0 T + 2\beta_2 \cos k_0 T - 2\beta_2 + \\
&\quad b_{01} \sin k_0 n_1 T + O(\alpha) \equiv 0, \\
&\psi_5/\alpha = \beta_4 \cos k_0 T - \beta_4 + \beta_5 \sin k_0 T + O(\alpha) \equiv 0, \\
&\psi_6/\alpha = -k_0 \beta_4 \sin k_0 T + k_0 \beta_5 \cos k_0 T - k_0 \beta_5 + O(\alpha) \equiv 0, \\
&\psi_4/\alpha = -2\beta_1 \cos k_0 T + 2\beta_1 k_0 - 2\beta_2 \sin k_0 T - b_{11} k_0 n_1 \cos k_0 n_1 T + \\
&\quad b_{11} k_0 n_1 + O(\alpha) \equiv 0
\end{aligned} \tag{2.4}$$

Using the fact that the Jacobi integral exists for system (2.2), we can express β_6 in terms of $\beta_1, \ldots, \beta_5$. Therefore we can discard the last equation of (2.4). For the remaining five equations we have the Jacobian

$$\frac{D(\psi_1, \psi_2, \psi_3, \psi_5, \psi_6)}{D(\beta_1, \ldots, \beta_5)} = \frac{24\pi}{k_0(n_0-1)} \sin^4 \frac{\pi}{n_0-1}$$

which leads to the following theorem.

Theorem 1. If k_0 is a real number different from zero $n_0 k_1 + k_0 n_1 = k_0 k_1$ and the quantities $n_0 = p/q$ and $q/(p-q)$ are not integers $(p, q = \pm 1, \pm 2, \ldots)$, then system (2.2) has a formal solution in the form of series (2.3) with period $T = 2\pi k_0^{-1} q$.

The convergence of the solutions (2.3) can be proved using the method of majorants /3/. Let us rewrite system (2.2) in the form

$$du_k/d\tau = f_k(u_1, \ldots, u_6, \alpha), \quad k = 1, \ldots, 6 \tag{2.5}$$

assuming that $\xi = u_1$, $\xi' = u_2$, $\eta = u_3$, $\eta' = u_4$, $\zeta = u_5$, $\zeta' = u_6$.

We have

$$
\begin{aligned}
&u_i' = u_{i+1} \quad (i = 1, 3, 5), \quad u_2' = (1 + k_0)(k_0 + u_4)^2 + f\frac{\partial \varkappa}{\partial u_1} \\
&u_4' = -\frac{2u_2(k_0 + u_4)}{1 + u_1} + f\frac{\partial \varkappa}{\partial u_3}, \quad u_6' = f\frac{\partial \varkappa}{\partial u_5} \\
&\varkappa = \frac{m_0}{R} + \frac{m_1}{R_0}, \quad \frac{1}{R_0} = \frac{1}{A^{1/2} X_0 (1 + z/A)^{1/2}} \\
&A = 1 - 2\frac{\alpha}{X}\cos((k_0 - k_0 n_1)\tau + u_3) + \left(\frac{\alpha}{X_0}\right)^2 \\
&z = \left(\frac{\alpha}{X_0}\right)^2 (u_1^2 + u_5^2) + 2\frac{\alpha}{X_0}\left(-\cos(k_0(n_0 - 1)\tau + n_0) + \frac{\alpha}{X_0}\right)u_1 \\
&\frac{1}{R} = \frac{1}{\alpha}(1 + 2u_1 + u_1^2 + u_5^2)^{-1/2}
\end{aligned}
\tag{2.6}
$$

Let $\delta < 1$ and $|u_1| < \delta$, $|u_5| < \delta$, $\alpha/X_0 < 1/4$. Then

$$|z| < \frac{1}{8}\delta^2 + \frac{5}{8}\delta, \quad \frac{9}{16} \leqslant A \leqslant \frac{25}{16}$$

From this it follows that the function $1/R_0$ will be regular, provided that the following inequality holds:

$$2\delta^2 + 10\delta - 9 < 0$$

Thus if

$$|u_1| < \frac{\sqrt{43} - 5}{2}, \quad |u_5| < \frac{\sqrt{43} - 5}{2}, \quad \frac{\alpha}{X} < \frac{1}{4}$$

then the function $1/R_0$ is regular. We shall prove the reguarity of $1/R$ and $2u_1(k_0 + u_4)/(1 + u_1)$ in the same manner and under the same conditions.

This leads to the following theorem.

Theorem 2. If in the region

$$D = \{|u_i| < 1/2, \quad i = 1, \ldots, 6, \ \alpha/X_0 < 1/4, \ k_0 < 1\}$$

then the right-hand sides of (2.6) will be analytic functions bounded uniformly in τ.

We have, in the region D, the inequalities $|f_i(u_1, \ldots, u_6, \alpha)| < N$, $i = 1, \ldots, 6$, $N = 9/2 + [1 + f(m_0 + m_1)]/(7X_0)$, and this proves the theorem.

In order to prove the convergence of the series (2.3), we use the Cauchy theorem /4, p.45/. We expand each function $f_i(u_1, \ldots, u_6, \alpha)$ $(i = 1, \ldots, 6)$ in series. Since $|u_i| < 1/2$, we obtain from the Cauchy theorem

$$\left|\frac{1}{i_1! \ldots i_6!} \cdot \frac{\partial^{i_1 + \ldots + i_6} f(u_1, \ldots, u_6, \alpha)}{\partial^{i_1} u_1 \ldots \partial^{i_6} u_6}\right| < N\left(\frac{1}{2}\right)^{i_1 + \ldots + i_6}$$

Therefore a power series of special type, independent of

$$g(u) = g_k(u) = N\prod_{k=1}^{6}(1 - u_{k/2})^{-1}$$

will be a majorant for any of the series $f_i(u_1, \ldots, u_6, \alpha)$.

Since the right-hand sides of the equations

$$g_k' = g(y), \quad y_k(\tau_0) = b_0 \quad (k = 1, \ldots, 6) \quad (b_0 = \text{const})$$

are independent of k, it follows that their solutions $y_k(\tau)$ are given by a single power series $y = y(\tau)$ satisfying the equation

$$y' = N(1 - y/2)^{-6}, \quad y(\tau_0) = b_0$$

A straightforward integration yields

$$y = 2 - 2(b - 7N\tau/2)^{1/7}, \quad b = (1 - b_0/2)^7 + 7N\tau_0/2$$

The corresponding series converges when $|\tau| < 2b/7N$. Under the same conditions $|u_i(\tau)|$ will be even more convergent, and this proves the convergence.

Finally we obtain the following theorem.

Theorem 3. If $|\tau| < 2b/7N$, then series (2.3) converge absolutely in the region $\bar{D}$. The functions represented by these series are the solution of system (2.2).

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Translated by L.K.

PMM U.S.S.R.,Vol.50,No.5,pp.666-669,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

A METHOD OF CONSTRUCTING POLHODES OF AN INTERMEDIATE MOTION IN THE DYNAMICS OF A RIGID BODY*

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Asymptotic methods are used to construct the polhodes of an intermediate motion of a non-symmetric body about its centre of mass. The fundamental effects of this motion are governed by the action of the small external resistance of the medium, linear with respect to the angular velocity of rotation. Non-Eulerian motion is employed to construct the equations in osculating variables. A modification of the averaging procedure is proposed which makes it possible to obtain finite expressions for the polhodes of the intermediate motion. Results of the analysis of the intermediate motion and of the evolution of the polhodes of an Eulerian rotation of the body are given.

1. We consider the problem of the rapid motion of a non-symmetric rigid body about its centre of mass, whose basic effects are governed by the action due to the resistance of the surrounding medium, which is linear with respect to the angular velocity. Following /1/, we shall call the motions rapid, if the moment of external forces about a fixed point is small compared with the current value of the kinetic energy of rotation. We shall write the dynamic Euler equations, taking into account the specific features of the motion described earlier, in the form

$$Ap^{\cdot} + (C - B)\, qr = \varepsilon M_1\, (pqr,\, ABC,\, 123) \tag{1.1}$$

Here p, q, r are the projections of the angular velocity vector $\boldsymbol{\omega}$ onto the coordinate axes, A, B, C are the principal central moments of inertia of the body, ε is a small non-negative parameter, and $M_i\ (i = 1, 2, 3)$ are the components of the perturbing moment $\mathbf{M}$ where $\mathbf{M} = -I\boldsymbol{\omega}$, I is the matrix of the constant coefficients /2/ of resistance I_{ij} in associated axes $(i,\ j = 1, 2, 3)$. Henceforth we shall assume that $A > B > C$.

When studying the evolution of rapid motions of a rigid body about the centre of mass, we normally use the Euler-Poinsot motion as the generating motion obtained from Eqs.(1.1) for $\varepsilon = 0$, and we apply the method of varying the arbitrary Lagrange constants /1-5/ (of the generating solution). At the same time, the universal character of the Lagrange's method /5/ which can be used when choosing the unperturbed motion arbitrarily, makes it possible to carry out the investigation using motions resembling that described by Eqs.(1.1) more closely than the Eulerian motion. Such motions, which were first encountered in classical celestial mechanics, have become particularly valuable in connection with constructing the theory of the motion of artificial celestial bodies, and are called intermediate motions, while the corresponding trajectories are called intermediate orbits /3, 5, 5/.

The problem of constructing the trajectories (polhodes and herpolhodes) of the intermediate motion of a rigid body was discussed in /3/. The method involves taking into account the most significant special features of the rotational motion in such a manner that the corresponding equations can be integrated in closed form. The present paper gives a method of constructing the polhodes of the intermediate motion, taking into account the small forces opposing the rotation of the body.

*Prikl.Matem.Mekhan.,50,5,859-863,1986